Exam "Inleiding Theoretische Natuurkunde"

Tuesday April 12 2011 from 9.00 until 12.00 uur.

1. Black holes

In the presence of a black hole, space-time is curved into a Schwarzschild metric:

$$d\tau^2 = (1 - \frac{2M}{r})\, dt^2 - \frac{1}{(1 - \frac{2M}{r})} dr^2 \,, \tag{1}$$

where we have only given the temporal and radial components. Consider a black hole with a mass $M = 5$ km. A space station is located at a distance of $r = 20$ km from the black hole. Its crew includes twins, Alice and Bob.

a) Alice wants to further investigate the curvature of space-time and descends to a distance of $r = 15$ km. After spending some time here and going back to the space station, will she be younger than / as old as / older than Bob (who has remained on the space station). Explain your answer.

b) Suppose Alice remains for a full year at $r = 15$ km before returning to the space station, what will be the age difference between the twins?

c) Alice and Bob decide to measure the distance between them by lowering a rope from $r = 20$ km to $r = 15$ km. Will the distance as measured by the rope be more than / equal to / less than 5 km? Explain your answer.

2. Dirac equation

a) Suppose an electron is in a state described by the spinor

$$\chi = A \begin{bmatrix} 3i \\ 4 \end{bmatrix} .$$

Determine the normalization constant A. What are the probabilities of getting $\pm\hbar/2$ if you measure S_z?

b) The Dirac equation reads

$$i\hbar \frac{\partial \psi}{\partial t} = -i\hbar c\, \boldsymbol{\alpha} \cdot \nabla \psi + mc^2 \beta\, \psi \,,$$

where α^1, α^2, α^3, and β are 4×4 matrices. Define $\gamma^0 \equiv -i\beta$, $\gamma^i \equiv -i\beta\alpha^i = \gamma^0\alpha^i$. Show that the Dirac equation rewritten in manifestly Lorentz covariant form is:

$$\left(\gamma^\mu \frac{\partial}{\partial x^\mu} + \frac{mc}{\hbar}\right) \psi = 0 \,.$$

Hint: Multiply from the left by $-\beta/\hbar c$ and use $\beta^2 = I$.

c) Explain what the Dirac sea is, and why Dirac was led to this concept because of the negative-energy solutions of his equation. Illustrate the Dirac sea with a picture.

3. Electric field with extra dimensions

A point charge q in d spatial dimensions generates a radial electric field $\vec{E}(r)$.

a) Show that the condition $\partial_i E_j - \partial_j E_i = 0$ (with $\partial_i \equiv \partial/\partial x_i$) is satisfied by the ansatz $\vec{E} = -\vec{\nabla}\Phi$, with the gradient $\vec{\nabla} = (\partial_1, \partial_2, \ldots, \partial_d)$.

b) Given the explicit form of the electric field in d spatial dimensions

$$\vec{E}(r) = -\frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{d/2}} \frac{q}{r^{d-1}} \hat{r}\,, \tag{2}$$

with $\hat{r}$ the unit radial vector $\hat{r} = \vec{r}/r$, with $\vec{r} = (x_1, x_2, \ldots x_d)$, show that the potential Φ is given by

$$\Phi(r) = \frac{\Gamma\left(\frac{d}{2}-1\right)}{4\pi^{d/2}} \frac{q}{r^{d-2}}\,. \tag{3}$$

c) What are the units of the electric charge q? Write the result in terms of M (mass), L (lenght) and T (time). Recall that the force on a probe charge q' is $\vec{F} = q'\vec{E}$.

4. Drift in crossed electric and magnetic fields

Figure 1 shows the trajectory of a non-relativistic charged particle in uniform electric and magnetic fields, for $\mathbf{H}$ parallel to the z axis and $\mathbf{E}$ parallel to the y axis. The motion of the particle in the xy plane is the superposition of rotation around the z axis and drift in the x direction. Consider the following symmetry operations:

a) Time reversal $\mathcal{T} : t \to -t$,

b) Inversion $\mathcal{I} : \mathbf{x} \to -\mathbf{x}$,

c) Mirror $m_y : x \to x,\ y \to -y,\ z \to z$.

Apply the time reversal operation to the process shown in Fig. 1 and draw the resulting trajectory and directions of the electric and magnetic fields. Do the same for inversion and the m_y mirror. In each case indicate the direction of the drift.

Hint: The trajectories and fields transform in such a way that the equation of motion $m\frac{d^2\mathbf{x}}{dt^2} = q\mathbf{E} + \frac{q}{c}\left[\frac{d\mathbf{x}}{dt} \times \mathbf{H}\right]$, *where m and q are, respectively, the particle mass and charge, while c is the speed of light, remains invariant.*

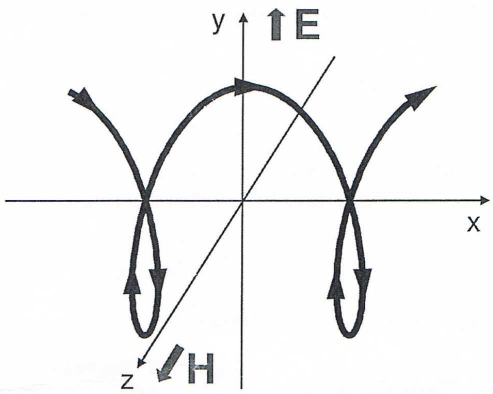

Figure 1: The curve, called trochoid, describing the motion of a charged particle in perpendicular electric and magnetic fields.